POWER SERIES COEFFICIENTS FOR PROBABILITIES IN FINITE CLASSICAL GROUPS

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Abstract. It is shown that a wide range of probabilities and limiting probabilities in finite classical groups have integral coefficients when expanded as a power series in q^{-1}. Moreover it is proved that the coefficients of the limiting probabilities in the general linear and unitary cases are equal modulo 2. The rate of stabilization of the finite dimensional coefficients as the dimension increases is discussed.

1. Introduction

Recently there has been interest in understanding the proportions of certain types of matrices over finite fields. For example an $n \times n$ matrix is called *separable* if its characteristic polynomial has no repeated roots, *semisimple* if its minimal polynomial has no repeated roots, and *cyclic* if its characteristic polynomial is equal to its minimal polynomial. Let $s_{\mathrm{M}(n,q)}$, $ss_{\mathrm{M}(n,q)}$ and $c_{\mathrm{M}(n,q)}$ respectively denote the probabilities that a random $n \times n$ matrix over $\mathbb{F}_q$ is separable, semisimple, or cyclic. Let $s_{\mathrm{GL}(n,q)}$, $ss_{\mathrm{GL}(n,q)}$, $c_{\mathrm{GL}(n,q)}$ denote the corresponding probabilities for a random element of $\mathrm{GL}(n, q)$.

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Motivated by questions in computational group theory [15], [18], Neumann and Praeger [16] proved that for $n \geq 2$,

$$1 - \frac{1}{(q^2-1)(q-1)} < c_{\mathrm{M}(n,q)} < 1 - \frac{1}{q^2(q+1)},$$

$$1 - \frac{q^2}{(q^2-1)(q-1)} - \frac{1}{2}q^{-2} - \frac{2}{3}q^{-3} < s_{\mathrm{M}(n,q)} < 1 - q^{-1} + q^{-2} + q^{-3}.$$

Let $s_{\mathrm{M}(\infty,q)}, ss_{\mathrm{M}(\infty,q)}, c_{\mathrm{M}(\infty,q)}, s_{\mathrm{GL}(\infty,q)}, ss_{\mathrm{GL}(\infty,q)}, c_{\mathrm{GL}(\infty,q)}$ be the limits of the proportions defined above as $n \to \infty$. Using generating function techniques, it was proved independently in [6] and [21] that

$$s_{\mathrm{M}(\infty,q)} = \prod_{r \geq 1}(1-q^{-r}), \quad c_{\mathrm{M}(\infty,q)} = (1-q^{-5})\prod_{r \geq 3}(1-q^{-r})$$

and

$$s_{\mathrm{GL}(\infty,q)} = 1 - q^{-1}, \quad c_{\mathrm{GL}(\infty,q)} = \frac{(1-q^{-5})}{(1+q^{-3})}.$$

Wall [21] obtained explicit estimates on the convergence to these limits. Concerning the semisimple limits, it was proved in [6] that

$$ss_{\mathrm{M}(\infty,q)} = \prod_{r \geq 1}(1-q^{-r}) \prod_{\substack{r \geq 1 \\ r \equiv 0, \pm 2 \pmod{5}}} (1-q^{-r+1})$$

and

$$ss_{\mathrm{GL}(\infty,q)} = \prod_{\substack{r \geq 1 \\ r \equiv 0, \pm 2 \pmod{5}}} \frac{(1-q^{-r+1})}{(1-q^{-r})}.$$

From these formulas, it is clear that the limiting probabilities, when expanded as a series in q^{-1}, have integer coefficients.

The above proportions have also been studied in the unitary, symplectic, and orthogonal groups [17], [10], [11]. Aside from the applications to computational group theory mentioned in the previous paragraph, there are applications to the study of derangements in group actions [8], [7] and to random generation of simple groups [9]. Formulas for $s_{\mathrm{G}(\infty,q)}, ss_{\mathrm{G}(\infty,q)}, c_{\mathrm{G}(\infty,q)}$ appear in [10] but are very complicated. For example

$$ss_{\mathrm{U}(\infty,q)} = (1+q^{-1}) \prod_{d \text{ odd}} A_{q,d}(1)^{\tilde{N}(d,q)} \prod_{d \geq 1} B_{q^2,d}(1)^{\tilde{M}(d,q)},$$

where

$$A_{q,d}(1) = (1-q^{-d})\left(1+\sum_{m\geq 1}\frac{1}{|\mathrm{U}(m,q^d)|}\right),$$

$$B_{q,d}(1) = (1-q^{-d})\left(1+\sum_{m\geq 1}\frac{1}{|\mathrm{GL}(m,q^d)|}\right),$$

and $\tilde{N}(d,q)$ and $\tilde{M}(d,q)$ enumerate certain sets of polynomials—see Section 2 for their definition.

A problem posed in [10] was to understand the integrality properties of the coefficients when the limits are expanded as series in q^{-1}. The separable case can be treated by adapting any of three quite different existing methods: Wall's combinatorial approach for $\mathrm{M}(d,q)$ and $\mathrm{GL}(d,q)$ [21], Lehrer's representation theoretic approach [13], or the topological approach of Lehrer and Segal [14]. However the semisimple case seems difficult by these approaches. In this paper we prove a general integrality result which can handle all of these cases.

Another result of this paper is a relation between the coefficients for limiting probabilities in the general linear and unitary groups. For instance comparing the above formulas for limiting general linear probabilities with formulas for unitary limiting probabilities in [10] one observes that

$$\begin{aligned}
s_{\mathrm{GL}(\infty,q)} &= 1-q^{-1}\\
s_{\mathrm{U}(\infty,q)} &= 1-q^{-1}-2q^{-3}+4q^{-4}-6q^{-5}+14q^{-6}-28q^{-7}+52q^{-8}-106q^{-9}+\cdots\\
c_{\mathrm{GL}(\infty,q)} &= 1-q^{-3}-q^{-5}+q^{-6}+q^{-8}-q^{-9}+\cdots\\
c_{\mathrm{U}(\infty,q)} &= 1-q^{-3}-q^{-5}+q^{-6}-2q^{-7}+3q^{-8}-5q^{-9}+\cdots\\
ss_{\mathrm{GL}(\infty,q)} &= 1-q^{-1}+q^{-3}-2q^{-4}+2q^{-5}-q^{-6}-q^{-7}+3q^{-8}-4q^{-9}+\cdots\\
ss_{\mathrm{U}(\infty,q)} &= 1-q^{-1}-q^{-3}+2q^{-4}-2q^{-5}+5q^{-6}-9q^{-7}+11q^{-8}-20q^{-9}+\cdots
\end{aligned}$$

These expansions suggest, and we prove, that quite generally the coefficients in the limiting general linear and unitary expansions are equal modulo 2. This is interesting because in the above cases there is a *simple* closed formula

for the general linear limits, but no such formula is known for the unitary limits.

We also establish an integrality result for the coefficients in the expansion as a series in q^{-1} of the probability that an element of a fixed group G satisfies certain properties; again the methods of [21], [13], [14] do not seem applicable at our level of generality. This result gives a slightly different approach to a question studied in [21], [13], [14]: how quickly the coefficients in the power series expansion of $s_{\mathrm{G}(d,q)}$ stabilize to the coefficients in the expansion of $s_{\mathrm{G}(\infty,q)}$. Results are also given for the cyclic case (studied in [21] for Lie algebras of type A). We complement these sharp results by giving a general approach to stabilization results which give reasonable bounds for a wide variety of cases.

The organization of the paper is as follows. Section 2 proves the integrality result for limiting coefficients, and the parity result relating the limiting coefficients in the general linear and unitary cases. Section 3 proves the integrality result for the case of a fixed group. Section 4 uses results from Section 3 to discuss the rate at which separable and cyclic coefficients stabilize to their limits, and proves a stabilization result for more general probabilities.

For many of out arguments we shall assume that the reader is familiar with cycle indices for finite matrix groups, and some of their applications. These have been developed in [12] and [19] and, for classical groups, [6]. For further developments, see [1],[2],[3],[4].

2. Integrality and parity of limiting coefficients

This section has two purposes. First, it will be shown that many fixed q large dimension limiting probabilities have integral coefficients when expanded as a power series in q^{-1}. The integrality result is established for the general linear, unitary, and symplectic groups. There is no need to state results for orthogonal groups since (as explained in the remark after the proof of Theorem 13) the arguments of [10] show that the corresponding

limiting probabilities are obtained from those of the symplectic group by multiplication by easily understood factors. Second, it will be proved that the limiting coefficients in the general linear and unitary cases are equal modulo 2. Our principal tools are the simple transforms in Lemmas 1 and 2, and the identities in Lemma 4, taken from [10].

Lemma 1. *Let $f(x)$ be the formal power series $1 + \sum_{i \geq 1} a_i x^i$. Then there exists a unique sequence (b_i) such that $f(x) = \prod_{i \geq 1}(1 - x^i)^{b_i}$. The sequence (b_i) consists entirely of integers if and only if every a_i is an integer.*

Proof. We define the exponents b_i recursively. Define $b_1 := -a_1$. Suppose that we have defined $b_1, \ldots b_n$, with the finite product

$$P_n(x) := \prod_{i=1}^{n}(1 - x^i)^{b_i}$$

being equal to $1 + \sum_{i \geq 1} c_i x^i$, and that we have done this in such a way that $c_i = a_i$ when $i \leq n$. Then we can force $P_{n+1}(x)$ to agree as far as the x^{n+1} coefficient by defining $b_{n+1} := c_{n+1} - a_{n+1}$. This is sufficient to prove existence and uniqueness of the exponents b_i. It is obvious that if every $b_i \in \mathbb{Z}$, then $f(x)$ has integer coefficients. The proof of the converse is by induction; suppose that all the $a_i \in \mathbb{Z}$. Then certainly $b_1 \in \mathbb{Z}$. Suppose that $b_1, \ldots, b_n$ are all integers. Then $\prod_{i=1}^{n}(1 - x^i)^{b_i}$ has integer coefficients, and so $c_{n+1} \in \mathbb{Z}$, and hence $b_{n+1} \in \mathbb{Z}$. $\square$

Lemma 2. *Let (a_i) be a sequence of even integers, and let $f(x)$ be the formal power series $1 + \sum_{i \geq 1} a_i x^i$. Then there exists a unique sequence (b_i) such that*

$$f(x) = \prod_{i \geq 1} \left(\frac{1 - x^i}{1 + x^i} \right)^{b_i}.$$

The sequence (b_i) consists entirely of integers if and only if every a_i is an even integer.

Proof. We first note that the function $\frac{1-x^i}{1+x^i}$, when expressed as a power series in x^i, has only even coefficients after the constant coefficient, and that the

coefficient of x^i is -2. As in the proof of Lemma 1 we define b_i recursively. First put $b_1 := -\frac{a_1}{2}$. Suppose that $b_1, \ldots, b_n$ have been defined; then define the partial product

$$P_n(x) := \prod_{i=1}^{n} \left(\frac{1-x^i}{1+x^i} \right)^{b_i}.$$

Let $P_n(x) = 1 + \sum_{i \geq 1} c_i x^i$, and suppose that $b_1, \ldots, b_n$ have been defined in such a way that $a_i = c_i$ when $i \leq n$. Then we can force $P_{n+1}(x)$ to agree with $f(x)$ as far as the x^{n+1} coefficient by defining $b_{n+1} := \frac{1}{2}(c_{n+1} - a_{n+1})$. Now since $P_n(x)$ consists of a finite product of power series with even coefficients, it follows that c_{n+1} is even. Since a_{n+1} is even by stipulation, it follows that b_{n+1} is an integer. $\square$

We define the following quantities, which count certain sets of polynomials. In this definition, we write μ for the arithmetic Möbius function.

Definition 3. *Let $e(q)$ be 1 if q is even, and 2 if q is odd.*

(a)

$$N(d,q) := \frac{1}{d} \sum_{a|d} \mu(a)(q^{\frac{d}{a}} - 1)$$

(b)

$$\widetilde{N}(d,q) := \begin{cases} \frac{1}{d} \sum_{a|d} \mu(a)(q^{\frac{d}{a}} + 1) & \text{if } d \text{ is odd,} \\ 0 & \text{if } d \text{ is even.} \end{cases}$$

(c)

$$\widetilde{M}(d,q) := \frac{1}{2} \left(N(d,q^2) - \widetilde{N}(d,q) \right)$$

(d)

$$N^*(d,q) := \begin{cases} \frac{1}{d} \sum_{\substack{a|d \\ a \text{ odd}}} \mu(a) \left(q^{\frac{d}{2a}} + 1 - e(q) \right) & \text{if } d \text{ is even,} \\ e(q) & \text{if } d = 1, \\ 0 & \text{if } d > 1,\ d \text{ odd.} \end{cases}$$

(e)

$$M^*(d,q) := \frac{1}{2} \Big(N(d,q) - N^*(d,q) \Big)$$

The following lemma brings together several identities from [10], namely Lemma 1.3.10 part (b), Lemma 1.3.14 parts (a) and (d), and Lemma 1.3.17 parts (a), (c), and (e).

Lemma 4. *Let* $e(q)$ *be* 1 *if* q *is even, and* 2 *if* q *is odd. Suppose that* $|x| < q^{-1}$. *Then*

(a)

$$\prod_{d\geq 1}(1-x^d)^{N(d,q)} = \frac{1-qx}{1-x}$$

(b)

$$\prod_{d \text{ odd}}(1-x^d)^{\widetilde{N}(d,q)}\prod_{d\geq 1}(1-x^{2d})^{\widetilde{M}(d,q)} = \frac{1-qx}{1+x}$$

(c)

$$\prod_{d \text{ odd}}\left(\frac{1-x^d}{1+x^d}\right)^{\widetilde{N}(d,q)} = \frac{(1-x)(1-qx)}{(1+x)(1+qx)}$$

(d)

$$\prod_{d\geq 1}(1-x^d)^{N^*(2d,q)}\prod_{d\geq 1}(1-x^d)^{M^*(d,q)} = \frac{1-qx}{(1-x)^{e(q)}}$$

(e)

$$\prod_{d\geq 1}\left(\frac{1-x^d}{1+x^d}\right)^{N^*(2d,q)} = \frac{1-qx}{(1-x)^{e(q)-1}}$$

(f)

$$\prod_{d\geq 1}(1-x^d)^{N^*(2d,q)}\prod_{d\geq 1}(1+x^d)^{M^*(d,q)} = \frac{1-qx^2}{(1-x)^{e(q)-1}(1+x)^{e(q)}}$$

We are now able to state and prove our first results on integrality of power series coefficients in Lemmas 5 and 6. Recall that an infinite product $\prod_n(1+r_n)$ is said to converge absolutely if $\prod_{n=1}^N(1+|r_n|)$ converges, and that $\prod_n(1+r_n)$ converges absolutely over a domain D in the complex plane if and only if $\sum_n |r_n|$ converges over D.

Lemma 5. *Let $r > q^{-1}$, and let (a_i) be a series of integers such that the product*

$$P(x) := \prod_{d\geq 1}\left(1+\sum_{i\geq 1} a_i x^{di}\right)^{N(d,q)}$$

converges absolutely whenever $|x| < r$. Then $P(q^{-1})$ has a power series expansion in q^{-1} with integer coefficients. Furthermore, if we define

$$Q(x) := \prod_{d \text{ odd}}\left(1+\sum_{i\geq 1} a_i x^{di}\right)^{\widetilde{N}(d,q)} \prod_{d\geq 1}\left(1+\sum_{i\geq 1} a_i x^{2di}\right)^{\widetilde{M}(d,q)}$$

and

$$R(x) := \prod_{d\geq 1}\left(1+\sum_{i\geq 1} a_i x^{di}\right)^{N^*(2d,q)} \prod_{d\geq 1}\left(1+\sum_{i\geq 1} a_i x^{di}\right)^{M^*(d,q)}$$

then the expansions of $Q(q^{-1})$ and $R(q^{-1})$ also have integer coefficients.

Proof. By Lemma 1, there exists a unique integer series (b_i) which satisfies $1+\sum_{i\geq 1} a_i x^i = \prod_{i\geq 1}(1-x^i)^{b_i}$. We write

$$\begin{aligned} P(x) &= \prod_{d\geq 1}\prod_{i\geq 1}(1-x^{di})^{b_i N(d,q)} \\ &= \prod_{i\geq 1}\left(\prod_{d\geq 1}(1-x^{di})^{N(d,q)}\right)^{b_i}. \qquad (1) \end{aligned}$$

Now for a given i, the product $\prod_{d\geq 1}(1-x^{di})^{N(d,q)}$ converges only when $|x^i| < q^{-1}$. Since $P(x)$ is absolutely convergent when $|x| < r$, it follows that $b_i = 0$ for any i such that $|r|^i > q^{-1}$. Hence every term $\prod_{d\geq 1}(1-x^{di})^{N(d,q)}$ which is present (i.e. has non-zero exponent b_i) in the product (1) converges when $|x| < r$. By part (a) of Lemma 4, we obtain

$$P(x) = \prod_{i\geq 1}\left(\frac{1-qx^i}{1-x^i}\right)^{b_i}.$$

In particular, this identity is valid when $x = q^{-1}$, which shows that the expansion of $P(q^{-1})$ in powers of q^{-1} has integer coefficients.

The products $Q(x)$ and $R(x)$ may be treated in exactly the same way, except that instead of appealing to part (a) of Lemma 4, we use part (b) for $Q(x)$ and part (d) for $R(x)$. Note also that absolute convergence of $P(x)$ for $|x| < r$ implies absolute convergence of $Q(x)$ and $R(x)$ for $|x| < r$, by the criterion mentioned before the statement of the lemma. $\square$

Lemma 5 is sufficient to deal with limiting probabilities in general linear groups. For the other classical groups, we require the following complementary result.

Lemma 6. *Let $r > q^{-1}$, and let (a_i) be a sequence of even integers such that the product*

$$\prod_{d\geq 1}\left(1+\sum_{i\geq 1} a_i x^{di}\right)^{N(d,q)}$$

converges absolutely for $|x| < r$. Define

$$\begin{aligned} A(x) &:= \prod_{d \text{ odd}}\left(1+\sum_{i\geq 1} a_i x^{di}\right)^{\widetilde{N}(d,q)} \\ B(x) &:= \prod_{d\geq 1}\left(1+\sum_{i\geq 1} a_i x^{di}\right)^{N^*(2d,q)}. \end{aligned}$$

Then the power series expansions in q^{-1} of $A(q^{-1})$ and $B(q^{-1})$ have integer coefficients.

Proof. By Lemma 2, we may write

$$1+\sum_{i\geq 1} a_i x^i = \prod_{i\geq 1}\left(\frac{1-x^i}{1+x^i}\right)^{b_i}.$$

Now we may proceed as in the proof of Lemma 5, making use of parts (c) and (e) of Lemma 4 for $A(x)$ and $B(x)$ respectively. $\square$

Lemma 6 leads us to consider the parity of the coefficients (a_i) of the power series lying within our infinite products. This approach turns out to be fruitful in terms of proving integrality of the coefficients of the expanded

power series, and will also lead to the somewhat unexpected result given in Theorem 15. The following lemma is useful in this respect.

Lemma 7. *Let (a_i) and (b_i) be sequences of integers. Then the power series expansion of*

$$\frac{1+\sum_i a_i x^i}{1+\sum_i b_i x^i}$$

has even coefficients (except for the constant coefficient) if and only if $a_i - b_i$ is even for all i.

Proof. In the ring $\mathbb{Z}[[x]]$ of formal power series in x with coefficients from $\mathbb{Z}$, let $\langle 2x \rangle$ be the principal ideal generated by $2x$. If $g(x)$ is invertible (i.e. has constant coefficient 1), then we observe that

$$f(x) - g(x) \in \langle 2x \rangle \iff g^{-1}(x)\Big(f(x) - g(x)\Big) \in \langle 2x \rangle \iff \frac{f(x)}{g(x)} \in 1 + \langle 2x \rangle.$$

This suffices to prove the lemma. $\square$

We may now bring together Lemmas 5 and 6 in the following way:

Lemma 8. *Let $r > q^{-1}$, and let (a_i) and (b_i) be sequences of integers such that $a_i - b_i$ is even for all i, and such that the products*

$$\prod_{d\geq 1}\left(1+\sum_{i\geq 1} a_i x^{di}\right)^{N(d,q)}$$

and

$$\prod_{d\geq 1}\left(1+\sum_{i\geq 1} b_i x^{di}\right)^{N(d,q)}$$

are absolutely convergent for $|x| < r$. Define

$$\begin{aligned} F(x) &:= \prod_{d \text{ odd}}\left(1+\sum_i a_i x^{di}\right)^{\widetilde{N}(d,q)} \prod_{d\geq 1}\left(1+\sum_i b_i x^{2di}\right)^{\widetilde{M}(d,q)} \\ G(x) &:= \prod_{d\geq 1}\left(1+\sum_i a_i x^{di}\right)^{N^*(2d,q)} \prod_{d\geq 1}\left(1+\sum_i b_i x^{di}\right)^{M^*(d,q)}. \end{aligned}$$

Then $F(q^{-1})$ and $G(q^{-1})$ have power series expansions in q^{-1} with integer coefficients.

Proof. We may rewrite $F(x)$ as

$$(2) \qquad F(x) = \prod_{d \text{ odd}} \left(\frac{1 + \sum_i a_i x^{di}}{1 + \sum_i b_i x^{di}} \right)^{\widetilde{N}(d,q)} \prod_{d \text{ odd}} \left(1 + \sum_i b_i x^{di} \right)^{\widetilde{N}(d,q)} \times \prod_{d \geq 1} \left(1 + \sum_i b_i x^{2di} \right)^{\widetilde{M}(d,q)}.$$

Now by Lemma 7, the quotient

$$\frac{1 + \sum_i a_i x^{di}}{1 + \sum_i b_i x^{di}}$$

expands with even coefficients except for the constant term. It follows from Lemma 6 that the first of the three infinite products in (2) expands with integer coefficients. And by Lemma 5, so do the second and third products taken together. This proves the result for $F(x)$; the proof for $G(x)$ is similar. □

From this point on we shall be working in full generality, rather than concentrating on the particular examples of cyclic, separable or semisimple elements. We work directly with generating functions derived from cycle indices. For background on cycle indices of finite classical groups, see [6]. The paper [19] is another useful reference and works out examples of cycle index calculations for $\mathrm{GL}(d,q)$ and $\mathrm{M}(d,q)$.

Definition 9. *Let Λ be a (possibly infinite) set of partitions of positive integers. Let α be an element of a finite classical group* G*. For each monic irreducible polynomial f over $\mathbb{F}_q$, define $\lambda_f(\alpha)$ to be the partition whose parts are $a_1 \dots a_n$, where $f^{a_1}, \dots, f^{a_n}$ are the powers of f amongst the elementary divisors of α.*

(1) *If $\mathrm{G} \in \{\mathrm{GL}, \mathrm{U}\}$, we say α is of Λ-type if the partitions $\lambda_f(\alpha)$ are all either empty or in Λ.*
(2) *If $\mathrm{G} = \mathrm{Sp}$, we say α is of Λ-type if the partitions $\lambda_f(\alpha)$ for $f \neq z \pm 1$ are all either empty or in Λ, and $\lambda_{z\pm1}(\alpha)$ are empty.*

Definition 10. *Define* $\Lambda_{\mathrm{G}(d,q)}$ *to be the probability that a randomly chosen element of* $\mathrm{G}(d,q)$ *is of* Λ*-type, and* $\Lambda_{\mathrm{G}(\infty,q)}$ *to be the limiting probability as* d *increases, the limit being taken only over even values of* d *if* $\mathrm{G} = \mathrm{Sp}$. *(The existence of this limit is explained in the proof of Theorem 13.)*

Remark: The requirement in Definition 9 that $\lambda_{z\pm1}$ should be empty in the symplectic case is for convenience. Limiting probabilities without this restriction are obtained by multiplying limiting probabilities with this restriction by a factor corresponding to $z-1$ and a factor corresponding to $z+1$. These factors are easily understood in any particular case.

We define quantities $C_{\mathrm{GL},\lambda}(q^d)$ and $C_{\mathrm{U},\lambda}(q^d)$ which appear when working with cycle indices. These quantities are sizes of certain centralizers, but we do not need this fact and shall define them by formulae.

Definition 11. *Let* λ *be a partition with* m_i *parts of size* i *for all* i, *and let* $k(\lambda) = 2\sum_{i<j} im_im_j + \sum_i (i-1)m_i^2$. *Then*

$$\begin{aligned} C_{\mathrm{GL},\lambda}(q^d) &:= q^{k(\lambda)d} \prod_i |\mathrm{GL}(m_i,q^d)| \\ &= q^{k(\lambda)d} \prod_i q^{d\binom{m_i}{2}} (q^{dm_i}-1)\cdots(q^d-1), \\ C_{\mathrm{U},\lambda}(q^d) &:= q^{k(\lambda)d} \prod_i |\mathrm{U}(m_i,q^d)| \\ &= q^{k(\lambda)d} \prod_i q^{d\binom{m_i}{2}} (q^{dm_i}-(-1)^{m_i})\cdots(q^d+1). \end{aligned}$$

The following lemma will be useful.

Lemma 12. (1) $\displaystyle\sum_{|\lambda|=n} \frac{1}{C_{\mathrm{GL},\lambda}(q^d)} = \frac{1}{q^{nd}(1-q^{-d})\cdots(1-q^{-nd})}$.

(2) $\displaystyle\sum_{|\lambda|=n} \frac{1}{C_{\mathrm{U},\lambda}(q^d)} = \frac{1}{q^{nd}(1+q^{-d})\cdots(1-(-1)^n q^{-nd})}$.

Proof. The first assertion may be found in [19]; it is a consequence of Fine and Herstein's count of nilpotent matrices [5]. For $d = 1$ the second assertion

follows from the first, since

$$C_{\mathrm{U},\lambda}(q) = (-1)^{|\lambda|} C_{\mathrm{GL},\lambda}(-q).$$

For general d replace q by q^d. $\square$

We are now in a position to establish a principal result of this section. As mentioned earlier and as explained after the proof of Theorem 13, there is no need to state results for orthogonal groups.

Theorem 13. *Let Λ be a set of partitions of positive integers. Then $\Lambda_{\mathrm{G}(\infty,q)}$ may be expressed as a power series in q^{-1} whose coefficients are integers if* $\mathrm{G} \in \{\mathrm{GL}, \mathrm{U}, \mathrm{Sp}\}$.

Proof. If Λ does not contain the unique partition of 1, it is not hard to show that $\Lambda_{\mathrm{G}(\infty,q)} = 0$ (this also follows from Theorem 25 in the case of GL). We shall therefore suppose throughout that Λ does contain this partition. Throughout the proof we use the notation that if $A(u) = \sum a_n u^n$ and $B(u) = \sum b_n u^n$, then $A(u) \ll B(u)$ means that $a_n \leq b_n$ for all n. We also assume familiarity with cycle indices of finite classical groups [6].

(1) Suppose $\mathrm{G} = \mathrm{GL}$. In this case we may specialize the general linear group cycle index to get

$$1 + \sum_{d \geq 1} \Lambda_{\mathrm{GL}(d,q)} u^d = \prod_{d \geq 1} \left(1 + \sum_{\lambda \in \Lambda} \frac{u^{d|\lambda|}}{C_{\mathrm{GL},\lambda}(q^d)} \right)^{N(d,q)}.$$

By applying part (a) of Lemma 4 with $x = q^{-1}u$, this may be written as $\frac{A(u)}{1-u}$ where

$$A(u) = (1 - q^{-1}u) \prod_{d \geq 1} \left[(1 - q^{-d}u^d) \left(1 + \sum_{\lambda \in \Lambda} \frac{u^{d|\lambda|}}{C_{\mathrm{GL},\lambda}(q^d)} \right) \right]^{N(d,q)}.$$

From part 1 of Lemma 12 and the fact that $(1) \in \Lambda$, it is not hard to see that

$$\begin{aligned}
&\prod_{d\geq 1}\left[(1-q^{-d}u^d)\left(1+\sum_{\lambda\in\Lambda}\frac{u^{d|\lambda|}}{C_{\mathrm{GL},\lambda}(q^d)}\right)\right]^{N(d,q)} \\
\ll &\prod_{d\geq 1}\left(1+\frac{u^d}{q^d(q^d-1)}+\sum_{n\geq 2}\frac{u^{nd}}{q^{nd}(1-q^{-d})\cdots(1-q^{-nd})}\right)^{N(d,q)} \\
\ll &\prod_{d\geq 1}\left(1+\frac{2u^d}{q^{2d}}+4\sum_{n\geq 2}\frac{u^{nd}}{q^{nd}}\right)^{N(d,q)}.
\end{aligned}$$

The last step used the fact from [16] that $\frac{1}{(1-q^{-1})\cdots(1-q^{-n})} \leq 4$ for all n and $q \geq 2$. Thus $\frac{A(u)}{1-u}$ is analytic in an open disc of radius $q^{\frac{1}{2}}$ except for a simple pole at $u = 1$. It follows that $\Lambda_{\mathrm{GL}(\infty,q)}$ is equal to the residue at that pole, which is

$$(1-q^{-1})\prod_{d\geq 1}\left[(1-q^{-d})\left(1+\sum_{\lambda\in\Lambda}\frac{1}{C_{\mathrm{GL},\lambda}(q^d)}\right)\right]^{N(d,q)}.$$

We can certainly find integers (a_i) such that

$$(1-q^{-d})\left(1+\sum_{\lambda\in\Lambda}\frac{1}{C_{\mathrm{GL},\lambda}(q^d)}\right) = 1+\sum_{i\geq 1}a_i q^{-di}.$$

An argument similar to that of the previous paragraph shows that the product

$$F(x) := \prod_{d\geq 1}\left(1+\sum_{i\geq 1}a_i x^{di}\right)^{N(d,q)}$$

converges absolutely for $|x| < q^{-\frac{1}{2}}$, and hence we may appeal to Lemma 5 to show that $F(q^{-1})$ has integer coefficients in its expansion; since $\Lambda_{\mathrm{GL}(\infty,q)} = (1-q^{-1})F(q^{-1})$, this is enough to prove this case of the theorem.

(2) Suppose next that $\mathrm{G} = \mathrm{U}$. By specializing the cycle index of $\mathrm{U}(d, q)$, we obtain the identity

$$1 + \sum_{d \geq 1} \Lambda_{\mathrm{U}(d,q)} u^d$$

$$= \prod_{d \text{ odd}} \left(1 + \sum_{\lambda \in \Lambda} \frac{u^{d|\lambda|}}{C_{\mathrm{U},\lambda}(q^d)}\right)^{\widetilde{N}(d,q)} \prod_{d \geq 1} \left(1 + \sum_{\lambda \in \Lambda} \frac{u^{2d|\lambda|}}{C_{\mathrm{GL},\lambda}(q^{2d})}\right)^{\widetilde{M}(d,q)},$$

which by means of part (b) of Lemma 4 (with $x = q^{-1}u$) can be rewritten as

$$\frac{1 + q^{-1}u}{1 - u} \prod_{d \text{ odd}} \left[(1 - q^{-d}u^d)\left(1 + \sum_{\lambda \in \Lambda} \frac{u^{d|\lambda|}}{C_{\mathrm{U},\lambda}(q^d)}\right)\right]^{\widetilde{N}(d,q)} \times$$

$$\prod_{d \geq 1} \left[(1 - q^{-2d}u^{2d})\left(1 + \sum_{\lambda \in \Lambda} \frac{u^{2d|\lambda|}}{C_{\mathrm{GL},\lambda}(q^{2d})}\right)\right]^{\widetilde{M}(d,q)}.$$

Arguing as in the $\mathrm{G} = \mathrm{GL}$ case (but using both parts of Lemma 12), one sees that apart from the explicit simple pole at $u = 1$, this is analytic in an open disc of radius $q^{\frac{1}{2}}$. Hence the value of $\Lambda_{\mathrm{U}(\infty,q)}$ is equal to its residue at $u = 1$. This is equal to

$$(1 + q^{-1}) \prod_{d \text{ odd}} \left[(1 - q^{-d})\left(1 + \sum_{\lambda \in \Lambda} \frac{1}{C_{\mathrm{U},\lambda}(q^d)}\right)\right]^{\widetilde{N}(d,q)} \times$$

$$\prod_{d \geq 1} \left[(1 - q^{-2d})\left(1 + \sum_{\lambda \in \Lambda} \frac{1}{C_{\mathrm{GL},\lambda}(q^{2d})}\right)\right]^{\widetilde{M}(d,q)}.$$

We can find integer sequences (a_i) and (b_i) such that for all d,

$$(1 - q^{-d})\left(1 + \sum_{\lambda \in \Lambda} \frac{1}{C_{\mathrm{U},\lambda}(q^d)}\right) = 1 + \sum_{i \geq 1} a_i q^{-di},$$

$$(1 - q^{-d})\left(1 + \sum_{\lambda \in \Lambda} \frac{1}{C_{\mathrm{GL},\lambda}(q^d)}\right) = 1 + \sum_{i \geq 1} b_i q^{-di}.$$

Let us consider $C_{\mathrm{U},\lambda}(q^d)$ and $C_{\mathrm{GL},\lambda}(q^d)$ as polynomials in q. It is clear from the definitions of these quantities that the difference of the coefficients of these two polynomials is even for any given power

of q. It follows easily that the difference of the reciprocals $C_{\mathrm{U},\lambda}(q^d)^{-1}$ and $C_{\mathrm{GL},\lambda}(q^d)^{-1}$, when expanded as a power series in q^{-1}, will have even coefficients, and hence that $a_i - b_i$ is even for all i. Define

$$F(x) = \prod_{d \text{ odd}} \left(1 + \sum_{i \geq 1} a_i x^{di}\right)^{\widetilde{N}(d,q)} \prod_{d \geq 1} \left(1 + \sum_{i \geq 1} b_i x^{2di}\right)^{\widetilde{M}(d,q)}.$$

As in the case G = GL, both factors in the product $F(x)$ converge absolutely when $|x| < q^{-\frac{1}{2}}$. We now invoke Lemma 8, which tells us that the expansion of $F(q^{-1})$ in powers of q^{-1} has integer coefficients. But then this is also true for $\Lambda_{\mathrm{U}(\infty,q)}$, which is equal to $(1+q^{-1})F(q^{-1})$.

(3) Suppose G = Sp. Specializing the cycle index of $\mathrm{Sp}(2d,q)$ gives the identity

$$1 + \sum_{d \geq 1} \Lambda_{\mathrm{Sp}(2d,q)} u^d =$$
$$\prod_{d \geq 1} \left(1 + \sum_{\lambda \in \Lambda} \frac{u^{d|\lambda|}}{C_{\mathrm{U},\lambda}(q^d)}\right)^{N^*(2d,q)} \prod_{d \geq 1} \left(1 + \sum_{\lambda \in \Lambda} \frac{u^{d|\lambda|}}{C_{\mathrm{GL},\lambda}(q^d)}\right)^{M^*(d,q)}.$$

Using part (d) of Lemma 4 with $x = q^{-1}u$, and arguing as in the previous cases, one deduces that $\Lambda_{\mathrm{Sp}}(\infty, q)$ is equal to

$$(1-q^{-1})^{e(q)} \prod_{d \geq 1} \left[(1-q^{-d}) \left(1 + \sum_{\lambda \in \Lambda} \frac{1}{C_{\mathrm{U},\lambda}(q^d)}\right)\right]^{N^*(2d,q)} \times$$
$$\prod_{d \geq 1} \left[(1-q^{-d}) \left(1 + \sum_{\lambda \in \Lambda} \frac{1}{C_{\mathrm{GL},\lambda}(q^d)}\right)\right]^{M^*(d,q)}.$$

The remainder of the argument is similar to the previous cases.

□

Remark: As mentioned in the introduction, the limiting orthogonal probabilities are simple functions of the limiting symplectic probabilities. If $(1) \notin \Lambda$, it is not hard to show that all limiting probabilities are 0 (for

instance one could use an argument similar to that of Theorem 25). To handle the case $(1) \in \Lambda$, we extend an idea from [10] for the cases of separable, cyclic, and semisimple matrices. Suppose for example that the dimension of the space is even and that $\lambda_{z\pm 1}$ are empty. Then if one considers the 0-dimensional space to be of positive type, the sum of the cycle indices for the positive and negative type orthogonal groups is equal to the cycle index of the symplectic groups. Thus $\Lambda_{O^+(\infty,q)} + \Lambda_{O^-(\infty,q)} = \Lambda_{\mathrm{Sp}(\infty,q)}$. If one lets $X(u)$ denote the difference of the cycle indices for the positive and negative type orthogonal groups, then arguing as in [10] (or using Lemmas 2.6.1 and 3.7.2 of [20]) one deduces that

$$X(u) = \prod_{d\geq 1}\left(1+\sum_{\lambda\in\Lambda}\frac{(-1)^{|\lambda|}u^{d|\lambda|}}{C_{\mathrm{U},\lambda}(q^d)}\right)^{N^*(2d,q)}\prod_{d\geq 1}\left(1+\sum_{\lambda\in\Lambda}\frac{u^{d|\lambda|}}{C_{\mathrm{GL},\lambda}(q^d)}\right)^{M^*(d,q)}.$$

It follows that $X(u)$ is analytic in an open disc of radius $q^{1/2}$, which implies that $\Lambda_{O^+(\infty,q)} = \Lambda_{O^-(\infty,q)}$, and hence that both of these probabilities are equal to $\frac{\Lambda_{\mathrm{Sp}(\infty,q)}}{2}$. To prove the analyticity assertion about $X(u)$, one uses the fact that $(1) \in \Lambda$ and part (f) of Lemma 4 with $x = q^{-1}u$ to write $X(u)$ as

$$\begin{aligned}
&\prod_{d\geq 1}\left(1-\frac{u^d}{(q^d+1)}+\cdots\right)^{N^*(2d,q)}\prod_{d\geq 1}\left(1+\frac{u^d}{(q^d-1)}+\cdots\right)^{M^*(d,q)}\\
=\ &\frac{1-q^{-1}u^2}{(1-q^{-1}u)^{e(q)-1}(1+q^{-1}u)^{e(q)}}\prod_{d\geq 1}\left(\frac{1-\frac{u^d}{(q^d+1)}+\cdots}{(1-\frac{u^d}{q^d})}\right)^{N^*(2d,q)}\\
&\cdot\prod_{d\geq 1}\left(\frac{1+\frac{u^d}{(q^d-1)}+\cdots}{(1+\frac{u^d}{q^d})}\right)^{M^*(d,q)}.
\end{aligned}$$

Then one argues as in the G = GL case of the proof of Theorem 13.

As a corollary of Theorem 13, we answer one of the questions raised in [10]. Note that O refers to an orthogonal group on an odd dimensional space, and that $O^{\pm}$ refer to orthogonal groups on an even dimensional space.

Corollary 14. *The coefficients of powers of q^{-1} in the limiting probabilities $s_{\mathrm{G}(\infty,q)}, ss_{\mathrm{G}(\infty,q)}, c_{\mathrm{G}(\infty,q)}$ are integers for $\mathrm{G} \in \{\mathrm{GL}, \mathrm{U}, \mathrm{Sp}, O, O^+, O^-\}$ except for the cases*

(1) *$s_{O^\pm}$ in even characteristic,*
(2) *$c_{O^\pm}$ in odd or even characteristic,*
(3) *$ss_{O^\pm}$ in odd or even characteristic.*

For these three cases, the coefficients are half-integers.

Proof. This is clear from Theorem 13 and the formulas for limiting probabilities in [10]. □

The following theorem is a somewhat curious outcome of our study of parity. This relationship is interesting because as mentioned in the introduction, there are simple exact formulas for the limiting proportion of regular semisimple, cyclic, and semisimple matrices in the general linear case, but in the unitary case it is not even known which of these limiting proportions is a rational function of q.

Theorem 15. *Let Λ be a set of partitions of positive integers containing (1). Write*

$$\begin{aligned} \Lambda_{\mathrm{GL}(\infty,q)} &= 1 + \sum_{i \geq 1} a_i q^{-i} \\ \Lambda_{\mathrm{U}(\infty,q)} &= 1 + \sum_{i \geq 1} b_i q^{-i}. \end{aligned}$$

Then $a_i - b_i$ is even for all i.

Proof. Let us write

$$\begin{aligned} (1-q^{-d})\left(1 + \sum_{\lambda \in \Lambda} \frac{1}{C_{\mathrm{GL},\lambda}(q^d)}\right) &= 1 + \sum_{i \geq 1} v_i q^{-di}, \\ (1-q^{-d})\left(1 + \sum_{\lambda \in \Lambda} \frac{1}{C_{\mathrm{U},\lambda}(q^d)}\right) &= 1 + \sum_{i \geq 1} w_i q^{-di}. \end{aligned}$$

Then as we observed in the proof of Theorem 13, $v_i - w_i$ is even for all i, and

$$\begin{aligned}\Lambda_{\mathrm{U}(\infty,q)} &= (1+q^{-1}) \prod_{d \text{ odd}} \left(\frac{1+\sum_{i\geq 1} w_i q^{-di}}{1+\sum_{i\geq 1} v_i q^{-di}} \right)^{\widetilde{N}(d,q)} \times \\ & \prod_{d \text{ odd}} \left(1 + \sum_{i \geq 1} v_i q^{-di} \right)^{\widetilde{N}(d,q)} \prod_{d \geq 1} \left(1 + \sum_{i\geq 1} v_i q^{-2di} \right)^{\widetilde{M}(d,q)} .\end{aligned}$$

Following our usual procedure, we use Lemmas 1, 2, and 7 to transform this into the form

$$\begin{aligned}&(1+q^{-1}) \prod_{d \text{ odd}} \prod_{i \geq 1} \left(\frac{1-q^{-di}}{1+q^{-di}} \right)^{y_i \widetilde{N}(d,q)} \times \\ &\prod_{d \text{ odd}} \prod_{i\geq 1} (1-q^{-di})^{z_i \widetilde{N}(d,q)} \prod_{d\geq 1} \prod_{i \geq 1} (1-q^{-2di})^{z_i \widetilde{M}(d,q)}\end{aligned}$$

for integer series (y_i) and (z_i). Now we use parts (c) and (b) of Lemma 4 to obtain

$$\Lambda_{\mathrm{U}(\infty,q)} = (1+q^{-1}) \prod_{i \geq 1} \left[\frac{(1-q^{-i})(1-q^{1-i})}{(1+q^{-i})(1+q^{1-i})} \right]^{y_i} \prod_{i\geq 1} \left[\frac{1-q^{1-i}}{1+q^{-i}} \right]^{z_i} .$$

On the other hand, by the proof of Theorem 13 and part (a) of Lemma 4, we may write

$$\begin{aligned}\Lambda_{\mathrm{GL}(\infty,q)} &= (1-q^{-1}) \prod_{d \geq 1} \left(1 + \sum_i v_i q^{-di} \right)^{N(d,q)} \\ &= (1-q^{-1}) \prod_{i \geq 1} \left(\frac{1-q^{1-i}}{1-q^{-i}} \right)^{z_i} .\end{aligned}$$

Therefore

$$\frac{\Lambda_{\mathrm{U}(\infty,q)}}{\Lambda_{\mathrm{GL}(\infty,q)}} = \frac{1+q^{-1}}{1-q^{-1}} \prod_{i \geq 1} \left[\left(\frac{1-q^{-i}}{1+q^{-i}} \right)^{y_i + z_i} \left(\frac{1-q^{1-i}}{1+q^{1-i}} \right)^{y_i} \right],$$

the expansion of which has even coefficients (since $\frac{1-q^{-1}}{1+q^{-1}}$ does). The theorem now follows from Lemma 7. □

3. Integrality of Finite Dimensional Coefficients

This section proves an integrality result for the coefficients of probabilities in a finite classical group when expanded as a power series in q^{-1}. Here the group is fixed, so the result is non-asymptotic, which removes the need to deal with issues of convergence. For the special cases of regular semisimple elements in the setting of Lie algebras rather than Lie groups, the paper [13] gives interpretations of this result in terms of topology and representation theory of the Weyl group. The argument presented here is very much in the spirit of Section 2, but now one needs the following variations of Lemmas 1 and 2 which involve power series in two variables.

Lemma 16. *Let $f(u,q^{-1})$ be the formal power series $1+\sum_{1\le i,j} a_{i,j}u^iq^{-j}$. Then there exists a unique sequence $(b_{i,j})$ such that*

$$f(u,q^{-1}) = \prod_{1\le i,j}(1-u^iq^{-j})^{b_{i,j}}.$$

The sequence $(b_{i,j})$ consists of integers if and only if every $a_{i,j}$ is an integer.

Proof. The proof is nearly identical to that of Lemma 1, except that the exponents $b_{i,j}$ are defined by induction on $n := i+j$. Thus we define

$$\begin{aligned} b_{1,1} &:= -a_{1,1}, \\ b_{k,n-k} &:= -a_{k,n-k} + c_{k,n-k}, \end{aligned}$$

where $c_{k,n-k}$ is the coefficient of $u^kq^{-(n-k)}$ in the expansion of

$$\prod_{\substack{1\le i,j\\ i<k\\ j<n-k}} (1-u^iq^{-j})^{b_{i,j}}.$$

It is a straightforward matter to show that the $b_{i,j}$ satisfy the statements of the lemma. □

Lemma 17. *Let $(a_{i,j})$ be a sequence of even integers. Let $f(u,q^{-1})$ be the formal power series $1+\sum_{1\le i,j} a_{i,j}u^iq^{-j}$. Then there exists a unique sequence*

$(b_{i,j})$ such that

$$f(u,q^{-1}) = \prod_{1\le i,j} \left(\frac{1-u^iq^{-j}}{1+u^iq^{-j}}\right)^{b_{i,j}}.$$

The sequence $(b_{i,j})$ consists entirely of integers if and only if every $(a_{i,j})$ is an even integer.

Proof. The argument is a straightforward modification of Lemma 2, except that the $b_{i,j}$ are defined by induction on $i+j$, as in Lemma 16. □

Now the main result of this section can be proved.

Theorem 18. *Let Λ be a set of partitions of positive integers. Then $\Lambda_{\mathrm{G}(d,q)}$ may be expressed as a power series in q^{-1} whose coefficients are integers if $\mathrm{G} \in \{\mathrm{GL}, \mathrm{U}, \mathrm{Sp}\}$.*

Proof. The three cases are treated separately. For background on cycle indices of finite classical groups, see [6].

(1) Suppose that $\mathrm{G} = \mathrm{GL}$. The general linear group cycle index gives

$$1 + \sum_{d\ge 1} \Lambda_{\mathrm{GL}(d,q)} u^d = \prod_{d\ge 1} \left(1 + \sum_{\lambda\in\Lambda} \frac{u^{d|\lambda|}}{C_{\mathrm{GL},\lambda}(q^d)}\right)^{N(d,q)}.$$

By Lemma 16 and part (a) of Lemma 4, this can be rewritten as

$$\prod_{d\ge 1} \left(\prod_{1\le i,j} (1-u^{id}q^{-jd})^{b_{i,j}}\right)^{N(d,q)} = \prod_{1\le i,j} \left(\frac{1-u^iq^{1-j}}{1-u^iq^{-j}}\right)^{b_{i,j}}.$$

This implies the result in the $\mathrm{G} = \mathrm{GL}$ case.

(2) Suppose that $\mathrm{G} = \mathrm{U}$. Specializing the cycle index of the unitary groups gives that

$$\begin{aligned} &1 + \sum_{d\ge 1} \Lambda_{\mathrm{U}(d,q)} u^d \\ = &\prod_{d \text{ odd}} \left(1 + \sum_{\lambda\in\Lambda} \frac{u^{d|\lambda|}}{C_{\mathrm{U},\lambda}(q^d)}\right)^{\widetilde{N}(d,q)} \prod_{d\ge 1} \left(1 + \sum_{\lambda\in\Lambda} \frac{u^{2d|\lambda|}}{C_{\mathrm{GL},\lambda}(q^{2d})}\right)^{\widetilde{M}(d,q)}. \end{aligned}$$

Using the equation $C_{\mathrm{U},\lambda}(q) = (-1)^{|\lambda|} C_{\mathrm{GL},\lambda}(-q)$ and Lemma 16, this can be written as

$$\prod_{d \text{ odd}} \left(\prod_{1 \le i,j} (1 - \frac{(-1)^{i+j} u^{id}}{q^{jd}})^{b_{i,j}} \right)^{\tilde{N}(d,q)} \prod_{d \ge 1} \left(\prod_{1 \le i,j} (1 - \frac{u^{2id}}{q^{2jd}})^{b_{i,j}} \right)^{\tilde{M}(d,q)} .$$

By part (b) of Lemma 4, this is

$$\prod_{1 \le i,j} \left(\frac{1 - u^i q^{1-j}}{1 + u^i q^{-j}} \right)^{b_{i,j}} \prod_{d \text{ odd}} \left[\prod_{1 \le i,j} \left(\frac{(1 - (-1)^{i+j} u^{id} q^{-jd})}{(1 - u^{id} q^{-jd})} \right)^{b_{i,j}} \right]^{\tilde{N}(d,q)} .$$

When one writes

$$\prod_{1 \le i,j} \left(\frac{(1 - (-1)^{i+j} u^{id} q^{-jd})}{(1 - u^{id} q^{-jd})} \right)^{b_{i,j}} = 1 + \sum_{1 \le i,j} c_{i,j} u^i q^{-j},$$

the $c_{i,j}$ are clearly all even. Thus, applying Lemma 17 and then part (c) of Lemma 4 gives that

$$\begin{aligned}
& 1 + \sum_{d \ge 1} \Lambda_{\mathrm{U}(d,q)} u^d \\
= \quad & \prod_{1 \le i,j} \left(\frac{1 - u^i q^{1-j}}{1 + u^i q^{-j}} \right)^{b_{i,j}} \prod_{d \text{ odd}} \left[\prod_{1 \le i,j} \left(\frac{1 - u^{id} q^{-jd}}{1 + u^{id} q^{-jd}} \right)^{a_{i,j}} \right]^{\tilde{N}(d,q)} \\
= \quad & \prod_{1 \le i,j} \left(\frac{1 - u^i q^{1-j}}{1 + u^i q^{-j}} \right)^{b_{i,j}} \prod_{1 \le i,j} \left(\frac{(1 - u^i q^{-j})(1 - u^i q^{1-j})}{(1 + u^i q^{-j})(1 + u^i q^{1-j})} \right)^{a_{i,j}} .
\end{aligned}$$

This implies the result for the case $\mathrm{G} = \mathrm{U}$.

(3) Suppose that $\mathrm{G} = \mathrm{Sp}$. The argument is similar to the unitary case. Using Lemma 16, then part (d) of Lemma 4, followed by Lemma 17

and part (e) of Lemma 4, it follows that

$$
\begin{aligned}
& 1+\sum_{d\geq 1}\Lambda_{\mathrm{Sp}(2d,q)}u^d \\
= & \prod_{d\geq 1}\left(1+\sum_{\lambda\in\Lambda}\frac{u^{d|\lambda|}}{C_{\mathrm{U},\lambda}(q^d)}\right)^{N^*(2d,q)}\prod_{d\geq 1}\left(1+\sum_{\lambda\in\Lambda}\frac{u^{d|\lambda|}}{C_{\mathrm{GL},\lambda}(q^d)}\right)^{M^*(d,q)} \\
= & \prod_{d\geq 1}\left(\prod_{1\leq i,j}\left(1-\frac{(-1)^{i+j}u^{id}}{q^{jd}}\right)^{b_{i,j}}\right)^{N^*(2d,q)}\left(\prod_{1\leq i,j}(1-\frac{u^{id}}{q^{jd}})^{b_{i,j}}\right)^{M^*(d,q)} \\
= & \prod_{1\leq i,j}\left(\frac{1-u^iq^{1-j}}{(1-u^iq^{-j})^{e(q)}}\right)^{b_{i,j}}\prod_{d\geq 1}\left[\prod_{1\leq i,j}\left(\frac{1-\frac{(-1)^{i+j}u^{id}}{q^{jd}}}{1-\frac{u^{id}}{q^{jd}}}\right)^{b_{i,j}}\right]^{N^*(2d,q)} \\
= & \prod_{1\leq i,j}\left(\frac{1-u^iq^{1-j}}{(1-u^iq^{-j})^{e(q)}}\right)^{b_{i,j}}\prod_{d\geq 1}\left[\prod_{1\leq i,j}\left(\frac{1-u^{id}q^{-jd}}{1+u^{id}q^{-jd}}\right)^{a_{i,j}}\right]^{N^*(2d,q)} \\
= & \prod_{1\leq i,j}\left(\frac{1-u^iq^{1-j}}{(1-u^iq^{-j})^{e(q)}}\right)^{b_{i,j}}\prod_{1\leq i,j}\left(\frac{1-u^iq^{1-j}}{(1-u^iq^{-j})^{e(q)-1}}\right)^{a_{i,j}}.
\end{aligned}
$$

This completes the proof for the case $\mathrm{G}=\mathrm{Sp}$.

□

Remark: The case of the orthogonal groups is easily understood using the above approach. Suppose for instance that the dimension is even and that $\lambda_{z\pm 1}$ are empty. Then considering the 0 dimensional space to be of positive type, the sum of the cycle indices of the positive and negative type orthogonal groups is equal to the cycle index of the symplectic groups. So by Theorem 18, the coefficient of q^{-j} in the sum of the $O^+(2d,q)$ and $O^-(2d,q)$ probabilities is an integer. As in the remark after Theorem 13, let $X(u)$ denote the difference of the cycle indices for $O^+(2d,q)$ and $O^-(2d,q)$. Then as before $X(u)$ is equal to

$$\prod_{d\geq 1}\left(1+\sum_{\lambda\in\Lambda}\frac{(-1)^{|\lambda|}u^{d|\lambda|}}{C_{\mathrm{U},\lambda}(q^d)}\right)^{N^*(2d,q)}\prod_{d\geq 1}\left(1+\sum_{\lambda\in\Lambda}\frac{u^{d|\lambda|}}{C_{\mathrm{GL},\lambda}(q^d)}\right)^{M^*(d,q)},$$

which in the notation of the G = Sp case of Theorem 18, is equal to

$$\prod_{d\geq 1}\left(\prod_{1\leq i,j}(1-(-1)^j u^{id}q^{-jd})^{b_{i,j}}\right)^{N^*(2d,q)}\left(\prod_{1\leq i,j}(1-u^{id}q^{-jd})^{b_{i,j}}\right)^{M^*(d,q)}.$$

This is slightly different from the cycle index of the symplectic groups since the power of (-1) in the first factor is different, but the same argument as in the G = Sp case of Theorem 18 shows that the power series expansion in q^{-1} for the coefficient of u^d in $X(u)$ has integral coefficients. Thus the coefficient of q^{-j} in the $O^+(2d,q)$ and $O^-(2d,q)$ probabilities is a half-integer.

4. Rate of Stabilization of Coefficients

This section studies the question of how large d must be so that the coefficient of q^{-n} in $\Lambda_{\mathrm{G}(d,q)}$ is equal to the coefficient of q^{-n} in $\Lambda_{\mathrm{G}(\infty,q)}$. Subsection 4.1 obtains sharp results in the regular semisimple and cyclic cases, when the group in question is GL or U. For these groups it is well known that an element is regular semisimple if and only if it is separable. Then Subsection 4.2 uses themes from earlier sections of this paper to prove a general stabilization result; while not always sharp it is broadly applicable.

4.1. **Stabilization for Regular Semisimple and Cyclic Probabilities.** It should be noted that the regular semisimple case has been studied by several authors. Lehrer [13] obtained results in the setting of Lie algebras rather than Lie groups, but they were not sharp. Sharp results in types A, B, C for the Lie algebra case and for the GL case appear in [14] using topological methods. Wall [21] uses combinatorial techniques to obtain sharp results for GL and its Lie algebra for the regular semisimple case and cyclic case. The argument presented here has similarities to that of Wall [21], but seems different enough to record. Results are worked out for the general linear and unitary groups; similar methods will apply to the symplectic and orthogonal groups, but this is much more laborious.

Letting G denote GL or U, we use the notation that

$$s_{\mathrm{G}}(u,q) := 1 + \sum_{d\geq 1} u^d s_{\mathrm{G}(d,q)}$$

and

$$c_{\mathrm{G}}(u,q) := 1 + \sum_{d\geq 1} u^d c_{\mathrm{G}(d,q)}.$$

Here, as in the introduction, $s_{\mathrm{G}(d,q)}$ is the proportion of separable elements in $\mathrm{G}(d,q)$ and $c_{\mathrm{G}(d,q)}$ is the proportion of cyclic elements in $\mathrm{G}(d,q)$.

Proposition 19. *Let* $F(a,s) = \frac{1}{s}\sum_{r|s,a} \mu(r)(-1)^{s/r}\binom{s/r+a/r-1}{a/r}$. *Then*

$$s_{\mathrm{GL}}(u,q) = \frac{\prod_{a\geq 0}\prod_{s\geq 1}(1-u^s q^{1-s-a})^{F(a,s)}}{(1+\frac{u}{q-1})}.$$

Proof. From the cycle index of general linear groups [6],[19] ,

$$\begin{aligned}
\left(1+\frac{u}{q-1}\right) s_{\mathrm{GL}}(u,q) &= \prod_{d\geq 1}\left(1+\frac{u^d}{q^d-1}\right)^{\frac{1}{d}\sum_{r|d}\mu(r)q^{d/r}} \\
&= \exp\left(\sum_{d\geq 1}\frac{1}{d}\sum_{r|d}\mu(r)q^{d/r}\cdot\log\left(1+\frac{u^d}{q^d-1}\right)\right) \\
&= \exp\left(-\sum_{d\geq 1}\frac{1}{d}\sum_{r|d}\mu(r)q^{d/r}\sum_{i\geq 1}\frac{(-1)^i u^{id}}{iq^{id}(1-q^{-d})^i}\right).
\end{aligned}$$

Defining $s = ir$ and $t = d/r$, this becomes

$$\begin{aligned}
& \exp\left(-\sum_{s\geq 1}\sum_{t\geq 1}\frac{q^t u^{st}}{stq^{st}}\sum_{r|s}\mu(r)(-1)^{s/r}(1-q^{-rt})^{-s/r}\right) \\
= \ & \exp\left(-\sum_{s\geq 1}\sum_{t\geq 1}\frac{q^t u^{st}}{stq^{st}}\sum_{r|s}\mu(r)(-1)^{s/r}\sum_{b\geq 0}\binom{s/r+b-1}{b}q^{-rtb}\right).
\end{aligned}$$

Letting $a = rb$, this becomes

$$\begin{aligned}
& \exp\left(-\sum_{a\geq 0}\sum_{s\geq 1}\sum_{t\geq 1}\frac{(u^s q^{1-s-a})^t}{t}\frac{1}{s}\sum_{r|s,a}\mu(r)(-1)^{s/r}\binom{s/r+a/r-1}{a/r}\right) \\
= \ & \prod_{a\geq 0}\prod_{s\geq 1}(1-u^s q^{1-s-a})^{F(a,s)}.
\end{aligned}$$

□

As a consequence one has the following result, which is sharp in the sense that there are values of d (such as $d = 4$) for which the assertion would be false if the upper bound on n does not hold. Parts 2 and 3 of Theorem 20 are known from [21].

Theorem 20. (1) *The numbers*

$$F(a,s) := \frac{1}{s} \sum_{r|s,a} \mu(r)(-1)^{s/r} \binom{s/r + a/r - 1}{a/r}$$

are integers for all $a \geq 0, s \geq 1$.

(2) *The coefficients of q^{-n} in $s_{\mathrm{GL}(d,q)}$ and $s_{\mathrm{GL}(\infty,q)}$ are equal whenever $n \leq d-1$.*

(3) *The coefficients of q^{-n} in $c_{\mathrm{GL}(d,q)}$ and $c_{\mathrm{GL}(\infty,q)}$ are equal whenever $n \leq 2d$.*

(4) *The coefficients of q^{-n} in $s_{\mathrm{U}(d,q)}$ and $s_{\mathrm{U}}(\infty, q)$ are equal whenever $n \leq d-1$.*

(5) *The coefficients of q^{-n} in $c_{\mathrm{U}(d,q)}$ and $c_{\mathrm{U}(\infty,q)}$ are equal whenever $n \leq 2d$.*

Proof. For the first assertion, it follows from Theorem 18 that all coefficients $u^i q^{-j}$ in $(1 + \frac{u}{q-1}) s_{\mathrm{GL}}(u,q)$ are integers. Now consider the expression for $(1 + \frac{u}{q-1}) s_{\mathrm{GL}}(u,q)$ in Proposition 19. If some $F(a,s)$ were non-integral, let (a,s) be the smallest such, where smallest means to first compare the s coordinate, then if necessary the a coordinate. Then the coefficient of $u^s q^{1-s-a}$ in $(1 + \frac{u}{q-1}) s_{\mathrm{GL}}(u,q)$ would be non-integral, a contradiction.

For the second assertion, note by Möbius inversion that $F(0,1) = -1$, $F(0,2) = 1$ and that $F(0,m) = 0$ for $m \geq 2$. Thus by Proposition 19,

$$(1-u) s_{\mathrm{GL}}(u,q) = \frac{(1 - u^2 q^{-1})}{(1 + \frac{u}{q-1})} \prod_{a,s \geq 1} (1 - u^s q^{1-s-a})^{F(a,s)}.$$

The integrality of the $F(a,s)$ implies that q^{-d} divides the coefficient of u^{d+1} in $(1-u)s_{\mathrm{GL}}(u,q)$. This coefficient is $s_{\mathrm{GL}(d+1,q)} - s_{\mathrm{GL}(d,q)}$, which proves the result.

For the third assertion, it is proved in [21] that

$$c_{\mathrm{GL}(d+1,q)} - c_{\mathrm{GL}(d,q)} = q^{-d-1}\left[s_{\mathrm{GL}(d+1,q)} - s_{\mathrm{GL}(d,q)}\right].$$

Thus by the previous paragraph, q^{-2d-1} divides $c_{\mathrm{GL}(d+1,q)} - c_{\mathrm{GL}(d,q)}$ as a polynomial, which implies the result.

For the fourth assertion, note from Theorem 2.1.13 of [10] that

$$s_{\mathrm{U}}(u,q) = \frac{s_{\mathrm{GL}(u^2,q^2)}}{s_{\mathrm{GL}(-u,-q)}}.$$

From the expression for $(1-u)s_{\mathrm{GL}}(u,q)$ in the proof of the second assertion, it follows that $(1-u)s_{\mathrm{U}}(u,q)$ is equal to

$$(1-u^2q^{-1})\frac{\left(1+\frac{u}{q+1}\right)}{\left(1+\frac{u^2}{q^2-1}\right)}\prod_{a,s\geq 1}\left(\frac{(1-u^{2s}q^{2(1-s-a)})}{(1+(-1)^a u^s q^{1-s-a})}\right)^{F(a,s)}.$$

Thus Proposition 19 implies that q^{-d} divides the coefficient of u^{d+1} in $(1-u)s_{\mathrm{U}}(u,q)$. This coefficient is $s_{\mathrm{U}(d+1,q)} - s_{\mathrm{U}(d,q)}$, as desired.

For the fifth assertion, note from Theorem 2.1.10 of [10] that

$$c_{\mathrm{U}(d+1,q)} - c_{\mathrm{U}(d,q)} = (-q)^{-d-1}\left[s_{\mathrm{U}(d+1,q)} - s_{\mathrm{U}(d,q)}\right].$$

Thus by the previous paragraph, q^{-2d-1} divides $c_{\mathrm{U}(d+1,q)} - c_{\mathrm{U}(d,q)}$, which implies the result. □

4.2. **A General Stabilization Result.** This subsection gives an approach to finding the rate of stabilization of the finite dimensional coefficients to the limiting coefficients which is more general, in that it is effective for all Λ-types, though it does not give the sharpest possible results in all cases. We describe this approach only in the case of the groups $\mathrm{GL}(d,q)$, but it could be extended without difficulty to $\mathrm{G} \in \{\mathrm{U}, \mathrm{Sp}, O, O^{\pm}\}$.

We shall need the following extension of Lemma 16:

Lemma 21. *Let S be a subset of $\mathbb{N} \times \mathbb{N}$, closed under (componentwise) addition. Suppose that for integers $a_{i,j}$*

$$1 + \sum_{1 \leq i,j} a_{i,j} u^i q^{-j} = \prod_{1 \leq i,j} (1 - u^i q^{-j})^{b_{i,j}}.$$

Then $a_{i,j} = 0$ for all $(i,j) \notin S$ if and only if $b_{i,j} = 0$ for all $(i,j) \notin S$.

Proof. It is clear that the product

$$\prod_{\substack{1 \leq i,j \\ (i,j) \in S}} (1 - u^i q^{-j})^{b_{i,j}},$$

when expressed as a power series in u and q^{-1}, will only yield terms $u^i q^{-j}$ when $(i,j) \in S$. This is enough to prove one half of the double implication; the other half follows easily from induction on $n := i + j$, using the explicit construction of $b_{k,n-k}$ given in the proof of Lemma 16. □

Let Λ_0 be the set of all partitions of positive integers, and suppose that $\emptyset \neq \Lambda \subseteq \Lambda_0$. We find lower bounds for the rate of stabilization of the coefficients of the polynomials $\Lambda_{\mathrm{GL}(d,q)}$ as d increases. We use two similar methods, one for the case when $(1) \in \Lambda$, and the other for the case $(1) \notin \Lambda$. It is worth remarking that this particular distinction is intuitively reasonable; if $(1) \in \Lambda$, then all separable transformations are of Λ-type, and it follows that $\Lambda_{\mathrm{GL}(\infty,q)} \geq s_{\mathrm{GL}}(\infty, q) > 0$. But if $(1) \notin \Lambda$, it is easy to show—indeed our argument will show—that $\Lambda_{\mathrm{GL}(\infty,q)} = 0$. In the first case, our method will be to look at the difference $\Lambda_{\mathrm{GL}(d,q)} - \Lambda_{\mathrm{GL}(d-1,q)}$, and show that it is divisible (as a polynomial) by a particular power of q^{-1}. In the second case, we show that $\Lambda_{\mathrm{GL}(d,q)}$ itself is divisible by a power of q^{-1}.

For a non-empty partition λ, define $\Delta(\lambda)$ to be the degree of $C_{\mathrm{GL},\lambda}(q)$ as a polynomial in q. This degree may be expressed in several ways:

Lemma 22. (1) *Suppose that λ has m_i parts of size i for all i. Then $\Delta(\lambda) = 2\sum_{i<j} i m_i m_j + \sum_i i m_i^2$.*

(2) *Suppose that $n_i = \sum_{j \geq i} m_j$ for all i. Then $\Delta(\lambda) = \sum_i n_i^2$.*

(3) *Finally, suppose* λ *has parts* $a_1, \dots, a_k$*, where* $a_i \geq a_{i+1}$ *for all* i*. Then* $\Delta(\lambda) = \sum_i (2i-1)a_i = 2\sum_i ia_i - |\lambda|$

Proof. Definition 11 yields the first equation easily. The second follows from the first, via the observation that $\sum_{i<j} im_im_j = \sum_{k\geq 1}\sum_{k\leq i<j} m_im_j$. The third follows from the first by observing that, if $a_{s+1}, \dots a_{s+m_i}$ are the parts of size i, then $\sum_{k=1}^{m_i}(2(s+k)-1)a_{s+k} = 2ism_i + im_i^2$. Now use the fact that $s = \sum_{j>i} m_j$, and sum over i. □

It can be established easily (using any of the expressions for $\Delta(\lambda)$ above) that if $\#(\lambda)$ denotes the number of parts of λ, then

$$|\lambda| \leq \Delta(\lambda) \leq |\lambda|\#(\lambda) \leq |\lambda|^2.$$

Each of these inequalities may in fact be equality, and in each case this imposes a regular structure on λ; in particular, we remark that $|\lambda| = \Delta(\lambda)$ if and only if λ has a single part.

Define

$$T_\Lambda(u,q) := \sum_{\lambda\in\Lambda} \frac{u^{|\lambda|}}{C_{\mathrm{GL},\lambda}(q)}.$$

Then by the cycle index of $\mathrm{GL}(n,q)$ ([19], [6])

$$1 + \sum_{d\geq 1} \Lambda_{\mathrm{GL}(d,q)} u^d = \prod_{d\geq 1}\left(1 + T_\Lambda(u^d,q^d)\right)^{N(d,q)}. \tag{3}$$

Setting $\Lambda = \Lambda_0$ shows that

$$\prod_{d\geq 1}\left(1 + T_{\Lambda_0}(u^d,q^d)\right)^{N(d,q)} = \frac{1}{1-u}.$$

If we write Λ^c for the complement $\Lambda_0 \setminus \Lambda$, then we obtain from (3) the following equation:

$$1 + \sum_{d\geq 1}\left(\Lambda_{\mathrm{GL}(d,q)} - \Lambda_{\mathrm{GL}(d-1,q)}\right)u^d = \prod_{d\geq 1}\left(1 - \frac{T_{\Lambda^c}(u^d,q^d)}{1+T_{\Lambda_0}(u^d,q^d)}\right)^{N(d,q)}. \tag{4}$$

Here $\Lambda_{\mathrm{GL}(0,q)}$ is to be interpreted as 1.

Now $1 + T_{\Lambda_0}(u,q)$ may be written in the form $1 + \sum_{1\leq i\leq j} r_{i,j}u^iq^{-j}$. Its reciprocal can also be put into this form, since the modification of Lemma

16 in which all occurrences of $1 \le i,j$ are replaced by $1 \le i \le j$ is true. In fact it is shown in [19] that $(1+T_{\Lambda_0}(u,q))^{-1} = \prod_{r\ge 1}(1-uq^{-r})$. It follows that there are integers $a_{i,j}$ such that

$$1 - \frac{T_{\Lambda^c}(u,q)}{1+T_{\Lambda_0}(u,q)} = 1 + \sum_{2\le i,j} a_{i,j}u^i q^{-j}.$$

Note the condition $2 \le i,j$ on the index of summation, which is valid since Λ^c does not contain the partition (1). In fact it is easily shown that if $a_{i,j} \ne 0$, then there exists a partition λ in Λ^c such that $i \ge |\lambda|$, $j \ge \Delta(\lambda)$, and $j - i \ge \Delta(\lambda) - |\lambda|$. Let S be the set of all pairs (i,j) satisfying this condition. Then S is obviously closed under addition, and it follows from Lemma 21 that we can find integers $b_{i,j}$ such that

$$1 - \frac{T_{\Lambda^c}(u,q)}{1+T_{\Lambda_0}(u,q)} = \prod_{(i,j)\in S}(1-u^i q^{-j})^{b_{i,j}}.$$

Now by our usual argument, invoking Lemma 4, part (a),

$$\prod_{d\ge 1}\left(1 - \frac{T_{\Lambda^c}(u^d,q^d)}{1+T_{\Lambda_0}(u^d,q^d)}\right)^{N(d,q)} = \prod_{(i,j)\in S}\left(\frac{1-u^i q^{1-j}}{1-u^i q^{-j}}\right)^{b_{i,j}},$$

which may certainly be put into the form

$$\prod_{\substack{i,j\ge 1\\(i,j+1)\in S}}(1-u^i q^{-j})^{c_{i,j}}$$

for integers $c_{i,j}$.

Define $\sigma := \inf\{\, j/i \mid (i,j+1) \in S\}$. Then the set $\{(i,j) \mid j \ge i\sigma\}$ is closed under addition. By Lemma 21 it follows that, for some integers $e_{i,j}$, we may write

$$\prod_{d\ge 1}\left(1 - \frac{T_{\Lambda^c}(u^d,q^d)}{1+T_{\Lambda_0}(u^d,q^d)}\right)^{N(d,q)} = 1 + \sum_{\substack{i,j\ge 1\\ j\ge i\sigma}} e_{i,j}u^i q^{-j}.$$

Then from (4) above, it follows that for all d,

$$\Lambda_{\mathrm{GL}(d,q)} - \Lambda_{\mathrm{GL}(d-1,q)} = \sum_{j\ge d\sigma} e_{d,j}q^{-j},$$

and hence that, whenever $j < (d+1)\sigma$, the coefficient of q^{-j} in $\Lambda_{\mathrm{GL}(d,q)}$ has stabilized to the coefficient in the limit $\Lambda_{\mathrm{GL}(\infty,q)}$.

What is the ratio σ? Firstly, suppose that Λ^c contains at least one partition with a single part, and suppose that the smallest such partition is (k). Then $\Delta\,(\,(k)\,) = k$, and it is clear (since $\Delta(\lambda) > |\lambda|$ for partitions with more than one part) that $\sigma = \frac{k-1}{k}$. Suppose, on the other hand, that Λ^c contains no one-part partition. Then it is clear that $\frac{j-1}{i} \geq 1$ for any $(i,j) \in S$. But it is also clear that by taking i and j sufficiently large, we can make this ratio arbitrarily close to 1, and hence that $\sigma = 1$.

We summarize these conclusions in the following theorem:

Theorem 23. *Suppose $\{\,(1)\,\} \subseteq \Lambda \subseteq \Lambda_0$. Define*

$$\sigma := \begin{cases} 1 - \frac{1}{k} & \textit{if } k \textit{ is the size of the smallest one-part partition not in } \Lambda, \\ 1 & \textit{if } \Lambda \textit{ contains all one-part partitions.} \end{cases}$$

Then the coefficient of q^{-j} in $\Lambda_{\mathrm{GL}(d,q)}$ is equal to the coefficient in $\Lambda_{\mathrm{GL}(\infty,q)}$ whenever $j < (d+1)\sigma$.

We have the following applications to cyclic, separable and semisimple matrices:

Corollary 24.
(1) *The coefficients of q^{-j} in $c_{\mathrm{GL}(d,q)}$ and $c_{\mathrm{GL}(\infty,q)}$ are equal whenever $j \leq d$.*
(2) *The coefficients of q^{-j} in $s_{\mathrm{GL}(d,q)}$ and $s_{\mathrm{GL}(\infty,q)}$ are equal whenever $j \leq \frac{d}{2}$.*
(3) *The coefficients of q^{-j} in $ss_{\mathrm{GL}(d,q)}$ and $ss_{\mathrm{GL}(\infty,q)}$ are equal whenever $j \leq \frac{d}{2}$.*

Comparing with Theorem 20, these bounds are not sharp for the cases of $s_{\mathrm{GL}(d,q)}$ and $c_{\mathrm{GL}(d,q)}$. However it is not at all clear that the methods of [14], [21], or Theorem 20 can be adapted to the semisimple (or other) cases.

We now deal with the case when $(1) \notin \Lambda$. Let k be the smallest integer such that Λ contains a partition of k. For each $i \geq k$, define

$$\tau_i := \min\{\frac{\Delta(\lambda)}{|\lambda|} \mid \lambda \in \Lambda, |\lambda| \leq i\}.$$

Write

$$1 + T_\Lambda(u,q) = 1 + \sum_{2 \leq i,j} a_{i,j} u^i q^{-j}.$$

Note that $i, j \geq 2$ since $(1) \notin \Lambda$. Also it is not hard to see that $a_{i,j} = 0$ unless $j \geq i\tau_i$. Furthermore, since τ_i is weakly decreasing function of i, the set $S := \{(i,j) \mid j \geq i\tau_i,\ i,j \geq 2\}$ is additively closed. By Lemma 21, there are integers $b_{i,j}$ such that

$$1 + T_\Lambda(u,q) = \prod_{(i,j) \in S} (1 - u^i q^{-j})^{b_{i,j}}.$$

Now, proceeding as usual by means of part ((a)) of Lemma 4, we find that

$$\prod_{d \geq 1} \left(1 + T_\Lambda(u^d, q^d)\right)^{N(d,q)} = \prod_{(i,j) \in S} \left(\frac{1 - u^i q^{1-j}}{1 - u^i q^{-j}}\right)^{b_{i,j}},$$

which can be rewritten in the form

$$\prod_{(i,j+1) \in S} (1 - u^i q^{-j})^{c_{i,j}}. \tag{5}$$

Suppose $c_{i,j} \neq 0$. Then there is a partition λ_i such that $|\lambda_i| \leq i$, and $\frac{j+1}{i} \geq \frac{\Delta(\lambda_i)}{|\lambda_i|}$. But now

$$\frac{j}{i} \geq \frac{\Delta(\lambda_i)}{|\lambda_i|} - \frac{1}{i} \geq \frac{\Delta(\lambda_i)}{|\lambda_i|} - \frac{1}{|\lambda_i|} = \frac{\Delta(\lambda_i) - 1}{|\lambda_i|} \geq \sigma,$$

where $\sigma := \inf\{\frac{\Delta(\lambda)-1}{|\lambda|} \mid \lambda \in \Lambda\}$. The set $\{(i,j) \mid j \geq i\sigma,\ i,j \geq 1\}$ is additively closed, and so (5) may be written, by Lemma 21, in the form

$$\sum_{\substack{i,j \geq 1 \\ j \geq i\sigma}} e_{i,j} u^i q^{-j},$$

for integers $e_{i,j}$.

Now from (3) above, the expression at (5) is equal to $1 + \sum_d \Lambda_{\mathrm{GL}(d,q)} u^d$. It follows that

$$\Lambda_{\mathrm{GL}(d,q)} = \sum_{j \geq d\sigma} e_{d,j} q^{-j},$$

which suffices to prove the following theorem:

Theorem 25. *Suppose that $\emptyset \neq \Lambda \subseteq \Lambda_0$ and that $(1) \notin \Lambda$. Define*

$$\sigma := \inf\{\frac{\Delta(\lambda) - 1}{|\lambda|} \mid \lambda \in \Lambda\}.$$

Then whenever $j < d\sigma$, the coefficient of q^{-j} in $\Lambda_{\mathrm{GL}(d,q)}$ is 0.

The constant σ is likely to be fairly easy to calculate for most naturally arising sets Λ. If Λ contains one-part partitions, and (k) is the smallest, then $\sigma = 1 - \frac{1}{k}$. If Λ has no one-part partitions, then $\sigma \geq 1$.

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